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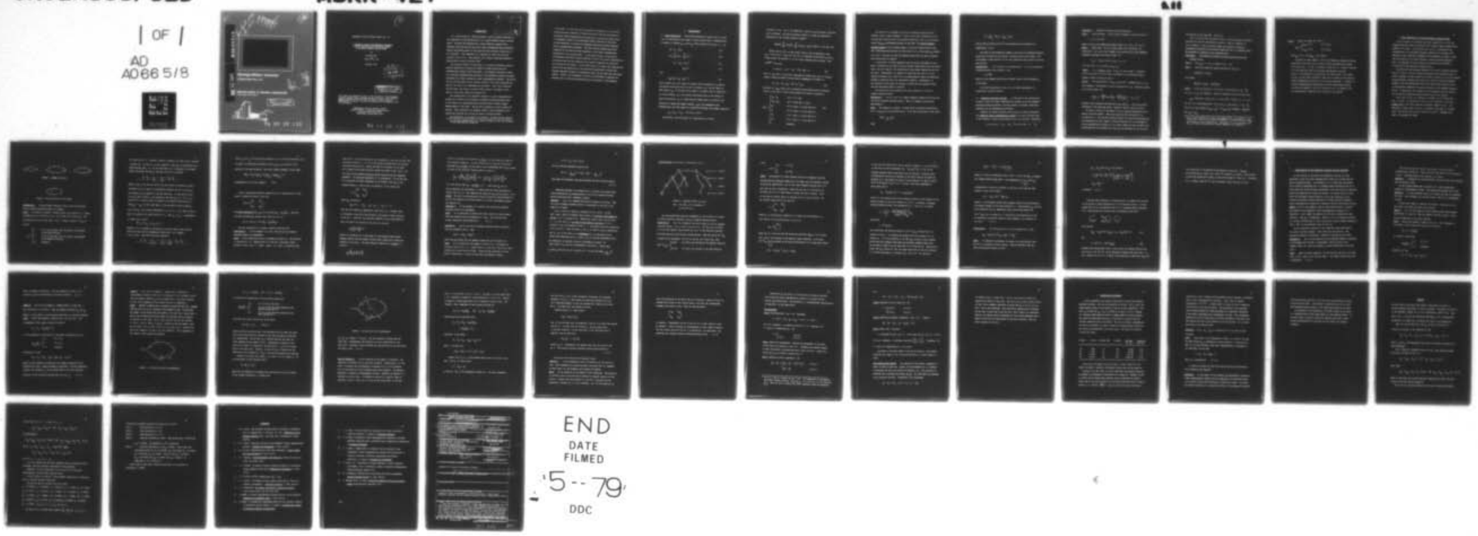

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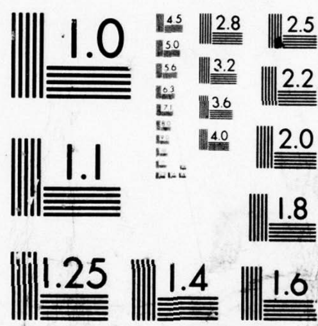
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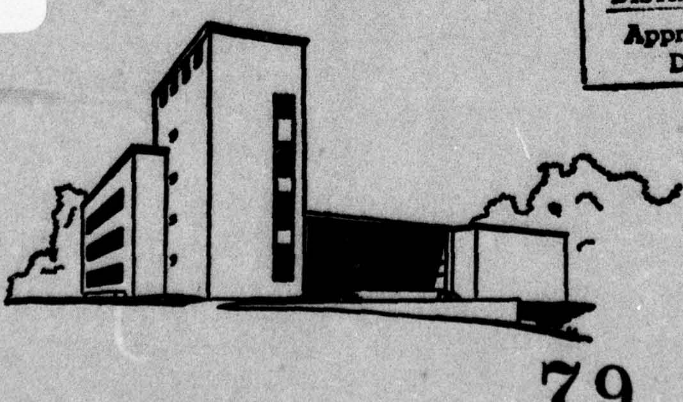

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Management Science Research Report No. 427

A PARAMETRIC LINEAR COMPLEMENTARITY TECHNIQUE FOR THE COMPUTATION OF EQUILIBRIUM PRICES IN A SINGLE COMMODITY SPATIAL MODEL

by

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and
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December 1978

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1. INTRODUCTION

In a recent paper [5], Glassey proposed a special-purpose algorithm for the computation of equilibrium prices in a single-commodity spatial economic model. The model was formulated as a convex quadratic program with a separable objective function and with certain transshipment-type constraints. Spatial equilibrium models of this type have been discussed intensively in the literature. (See Samuelson [15], Takayama and Judge [16] and more recently, Polito [14].) Unfortunately, the original algorithm proposed by Glassey fails to work in certain cases.*

The essential purpose of this paper is to present an efficient special-purpose algorithm for solving the quadratic program arising from the aforementioned economic equilibrium model. The proposed algorithm is based on an efficient implementation procedure of the parametric principal pivoting algorithm for solving a parametric linear complementarity problem. (See Cottle [2] for the description of the original version of the latter algorithm and Pang [9] for that of its implementation procedure.) This latter algorithm has been found very useful in various other contexts, see Kaneko [9,10], Pang [11,12] and Pang, Kaneko and Hallman [13].

The reformulation of the quadratic program as a linear complementarity problem to be solved by the parametric principal pivoting algorithm leads to the study of an interesting matrix associated with a simple digraph having weights on the nodes. Several basic properties of such a matrix are derived. These properties are extremely useful in the development of the specialized algorithm for solving the spatial equilibrium model.

The organization of the paper is as follows. The next section consists

* In a private communication [6], Glassey informed us that the statement of the algorithm was incomplete.

of two parts. In the first part, we review the formulation of the spatial equilibrium model addressed by Glassey and derive from this formulation an equivalent linear complementarity problem with a very special matrix which is intimately related to the underlying network structure of the model. In the second part, we review the parametric principal pivoting algorithm and its implementation procedure. The third section presents some basic properties of the special matrix appearing in the linear complementarity formulation of the economic model. In the fourth section, we specialize the parametric principal pivoting algorithm to solve this linear complementarity problem and show how the algorithm can be greatly simplified by exploiting the structure of the matrix. In the fifth section, we report some computational results with the application of the specialized algorithm in solving some randomly generated problems of considerably large size and draw some concluding remarks. Finally, in an appendix, we point out how Glassey's algorithm can sometimes fail to find the equilibrium prices and present a counterexample.

2. PRELIMINARIES

2.1 Model Formulation. The problem addressed by Glassey [5] is to find a set of equilibrium prices under a certain spatial model. Mathematically, it amounts to finding p_α, y_α and $x_{\alpha\beta}$ which satisfy for all $\alpha,\beta = 1,\ldots,N$

$$p_\alpha = a_\alpha - b_\alpha y_\alpha \tag{1a}$$

$$y_\alpha = \sum_{\beta=1}^{N} x_{\beta\alpha} - \sum_{\beta=1}^{N} x_{\alpha\beta} \tag{1b}$$

$$p_\alpha + c_{\alpha\beta} - p_\beta \geq 0 \tag{1c}$$

$$x_{\alpha\beta} \geq 0 \tag{1d}$$

and

$$x_{\alpha\beta}\,(p_\alpha + c_{\alpha\beta} - p_\beta) = 0. \tag{1e}$$

Here N denotes the total number of regions under consideration, p_α denotes the desired equilibrium price in the α-th region, y_α is the net import in that region, $x_{\alpha\beta}$ is the export from region α to region β and is referred to as a flow variable, a_α is the (given) equilibrium price in the absence of imports and exports, b_α is a given positive number which is related to the elasticity of supply and demand, finally, $c_{\alpha\beta}$ is the nonnegative unit transportation cost from region α to region β satisfying the triangle inequality

$$c_{\alpha\beta} \leq c_{\alpha\gamma} + c_{\gamma\beta} \quad \text{for all } \alpha,\ \beta \text{ and } \gamma\ .$$

Incidentally, the above model is a simplification of those discussed in [16]. Due to its simplicity, efficient special-purpose algorithm can be developed. As noted by Glassey, problem (1) forms the Kuhn-Tucker conditions for the (convex) quadratic program

$$\text{minimize} \sum_{\alpha=1}^{N} \left(\tfrac{1}{2} b_\alpha y_\alpha^2 + \sum_{\beta=1}^{N} c_{\alpha\beta} x_{\alpha\beta} - a_\alpha y_\alpha\right) \text{ subject to (1b) and (1d).}$$

Notice that by (1b), we may assume, with no loss of generality, that $x_{\alpha\alpha} = 0$ for all α. Using (1a) and (1b) to eliminate the variables p_α and y_α, we may easily cast problem (1) as the linear complementarity problem: find $z \in R^{N(N-1)}$ such that

$$q + Mz \geq 0, \quad z \geq 0 \quad \text{and} \quad z^T(q + Mz) = 0. \tag{2a}$$

Here $q = (q_k)$ and $z = (z_k)$ with k denoting the ordered pair (α,β) of distinct indices α and β are $N(N-1)$-vectors whose k-components are defined as

$$q_k = a_\alpha - a_\beta + c_{\alpha\beta} \quad \text{and} \quad z_k = x_{\alpha\beta}. \tag{2b}$$

Moreover, $M = (m_{k\ell})$ with k and ℓ denoting certain ordered pairs of distinct indices is the $N(N-1) \times N(N-1)$ matrix whose (k,ℓ)-entry is defined as

$$m_{k\ell} = \begin{cases} b_\alpha + b_\beta & \text{if } k = \ell = (\alpha,\beta) \\ -(b_\alpha + b_\beta) & \text{if } k = (\alpha,\beta) \text{ and } \ell = (\beta,\alpha) \\ b_\beta & \text{if } k = (\alpha,\beta),\ \ell = (\gamma,\beta) \text{ and } \alpha \neq \gamma \\ b_\alpha & \text{if } k = (\alpha,\beta),\ \ell = (\alpha,\gamma) \text{ and } \beta \neq \gamma \\ -b_\beta & \text{if } k = (\alpha,\beta),\ \ell = (\beta,\gamma) \text{ and } \alpha \neq \gamma \\ -b_\alpha & \text{if } k = (\alpha,\beta),\ \ell = (\gamma,\alpha) \text{ and } \beta \neq \gamma \\ 0 & \text{otherwise.} \end{cases} \tag{2c}$$

The matrix M is an example of an arc-arc weighted adjacency matrix. Formally, given a simple digraph $G = (V,A)$, with V and A denoting the sets of nodes and (directed) arcs of the graph respectively, and a set of positive scalars $\{b_\alpha\}_{\alpha \in N}$ representing weights on the nodes, the <u>arc-arc weighted adjacency matrix</u> of this weighted graph G is the real square matrix M such that, associated with each pair of arcs k and ℓ in A, is the entry $m_{k\ell}$ defined as in (2c). We point out that the digraph arising from the spatial economic equilibrium model is complete.

At this point, we should emphasize that the entire development of this paper, including all the results and the proposed algorithm for the equilibrium model, does not depend on the completeness of the simple digraph arising from the model. Consequently, our approach is applicable not only to the one treated by Glassey, but as well as to its generalization where the underlying digraph is arbitrary and is not necessarily complete. Nevertheless, in the remainder of this paper we continue to assume that the digraph arising from the equilibrium model is complete.

The proposition below identifies the first property of an arc-arc weighted adjacency matrix.

<u>Proposition 1</u>. Let $G = (V,A)$ be a simple (weighted) digraph with M denoting its arc-arc weighted adjacency matrix. Then M is symmetric and positive semi-definite.

<u>Proof</u>. The symmetry is obvious. To show that M is positive semi-definite, let $z = (z_k)_{k \in A}$ be an arbitrary vector. By an easy calculation, we may deduce

$$z^T M z = \sum_{\alpha \in V} b_\alpha y_\alpha^2$$

where

$$y_\alpha = \sum_{\beta \in P_\alpha} z_{(\beta,\alpha)} - \sum_{\beta \in S_\alpha} z_{(\alpha,\beta)}$$

with P_α and S_α denoting the sets of predecessors and successors of α respectively. Q.E.D.

The proof of the proposition suggests that an arc-arc weighted adjacency matrix may be related to the node-arc incidence matrix of the graph. This relationship is made explicit in the next proposition whose proof is easy and thus omitted.

Proposition 2. Let G and M be as in Proposition 1. If A is the node-arc incidence matrix of the digraph G, then

$$M = A^T DA$$

where D is the diagonal matrix whose diagonal entries are the weights b_α of the nodes.

As we shall demonstrate later, M is not always nonsingular, or equivalently, positive definite.

2.2. Parametric Principal Pivoting. As indicated in the introduction, we plan to solve the linear complementarity problem (2) by the parametric principal pivoting algorithm. For this purpose, we briefly review this algorithm and its implementation procedure.

Formally, for given n-vectors p and q with $p > 0$ and $n \times n$ matrix M, the parametric linear complementarity problem is to find, for each value of the parameter λ lying in some interval $[\underline{\lambda}, \infty)$, an n-vector z satisfying

$$q + \lambda p + Mz \geq 0, \quad z \geq 0 \quad \text{and} \quad z^T(q + \lambda p + Mz) = 0. \qquad (3)$$

Algorithm 1. **Parametric Principal Pivoting Algorithm.**

Step 0. (Initialization) Let the initial canonical system be given by

$$w = q + \lambda p + Mz$$

where $q + \lambda p \geq 0$ for sufficiently large values of λ. Put $j = 0$, $\lambda_0 = \infty$.

Step 1. (Ratio test) If $p \leq 0$ (this cannot occur initially), let $(w(\lambda), z(\lambda)) = (q + \lambda p, 0)$ for λ lying in the interval $[\underline{\lambda}, \lambda_j]$ and terminate. Otherwise, determine the critical index r by

$$\lambda_{j+1} = -q_r/p_r = \max\{-q_i/p_i : p_i > 0\}.$$

Put $(w(\lambda), z(\lambda)) = (q + \lambda p, 0)$ on $[\lambda_{j+1}, \lambda_j]$.

Step 2. (1 x 1 Diagonal pivot) If $m_{rr} = 0$, go to Step 3. Otherwise, pivot on m_{rr} and let w,z,p,q and M correspond to the resulting system. Replace j with j+1 and go to Step 1.

Step 3. (2 x 2 Block pivot) If $m_{ir} \geq 0$ for all i, redefine $\underline{\lambda}$ by $\underline{\lambda} = \lambda_{j+1}$ and terminate. The problem (3) has no solution for $\lambda \leq \underline{\lambda}$. Otherwise, define the critical index s by

$$-\frac{1}{m_{sr}}\left(q_s - \frac{p_s q_r}{p_r}\right) = \min\left\{\left(q_i - \frac{p_i q_r}{p_r}\right)\left(\frac{-1}{m_{ir}}\right) : m_{ir} < 0\right\}.$$

Perform a pivot operation successively on m_{rs} and m_{sr}. Let w,z,q,p and M correspond to the resulting system. Replace j with j+1 and go to Step 1.

We refer to Cottle [1] and Graves [7] for a detailed discussion of the theory of pivotal algebra. Under the condition that the initial M is a P-matrix (i.e. has positive principal minors) or is positive semi-definite, the algorithm above will compute a solution function $z(\lambda)$ to the parametric linear complementarity problem (3) in a finite number of steps, provided that some appropriate tie-breaking rule, like the one described in [7], has been incorporated in the algorithm. (See [2].)

The 2×2 block pivots described in Step 3 of the Algorithm are designed specially for positive semi-definite matrices. They are redundant if M is a P-matrix. In [11], the first author has proposed an efficient implementation procedure for actually carrying out the 1×1 diagonal pivots. The procedure is rephrased in the algorithm below.

Algorithm 2. **The Parametric Principal Pivoting Algorithm Using only Diagonal Pivots.**

Step 0. Let $\lambda_{old} = \infty$. Let $J = \emptyset$ and $I = \{1,\ldots,n\}$.

Step 1. Solve* the system of linear equations for $(\bar{q}_J, \bar{p}_J)$:

$$M_{JJ}(\bar{q}_J,\bar{p}_J) = (q_J,p_J)$$

and compute

$$(\bar{q}_I,\bar{p}_I) = (q_I,p_I) - M_{IJ}(\bar{q}_J,\bar{p}_J).$$

Step 2. If $\bar{p}_I \leq 0$ and $\bar{p}_J \geq 0$, set $(z(\lambda))_I = 0$ and $(z(\lambda))_J = -\bar{q}_J - \lambda\bar{p}_J$ for all $\lambda \leq \lambda_{old}$ and terminate. Otherwise determine the new critical value

$$\lambda_{new} = \max\{\max\{-\bar{q}_i/\bar{p}_i : \bar{p}_i > 0,\ i \in I\},\ \max\{-\bar{q}_j/\bar{p}_j : \bar{p}_j < 0,\ j \in J\}\}$$

and let k be a maximizing index. Put $(z(\lambda))_I = 0$ and $(z(\lambda))_J = -\bar{q}_J - \lambda\bar{p}_J$ for all λ in the interval $[\lambda_{new}, \lambda_{old}]$. If $\lambda_{new} \leq \underline{\lambda}$, terminate. Otherwise set $\lambda_{old} = \lambda_{new}$.

---

* If A is an $n \times m$ matrix and I and J are subsets of $\{1,\ldots,n\}$ and $\{1,\ldots,m\}$ respectively, then by A_{IJ} we mean the submatrix of A whose rows and columns are indexed by I and J respectively. Similarly, if q is an n-vector, then by q_I, we mean the subvector of q whose components are indexed by I.

Step 3. Update the index sets I and J:

$$J_{new} = \begin{cases} J_{old} \cup \{k\} & \text{if } k \notin J_{old} \\ J_{old} \setminus \{k\} & \text{if } k \in J_{old} \end{cases}$$

and $I_{new} = \{1,\ldots,n\} \setminus J_{new}$. Go to Step 1.

We point out three remarks. First, the parametric principal pivoting algorithm can be easily used to solve a linear complementarity problem of the form (2a). In fact, it suffices to choose $\underline{\lambda} = 0$. Second, Algorithm 2 is also applicable when M is positive semi-definite, provided that it is not necessary to perform the 2×2 block pivots. As a matter of fact, we shall show in Section 4 that this is indeed the case when the linear complementarity problem (2) is solved by Algorithm 1. Finally, if a linear complementarity problem with a positive semi-definite matrix is solved by Algorithm 1 and if all the pivots are 1×1 diagonal, then the algorithm must terminate with a solution to the problem.

3. BASIC PROPERTIES OF AN ARC-ARC WEIGHTED ADJACENCY MATRIX

In this section, we establish some basic properties of an arc-arc weighted adjacency matrix M associated with a general simple digraph having weights on its nodes. Our objectives are, first, to characterize the nonsingularity of an arbitrary principal submatrix of M in terms of the structure of the "associated subgraph" of the submatrix, and then, to show how the solution of the various systems of linear equations required in Step 1 of Algorithm 2 can be achieved by identifying the decendents of each node in the associated subgraph of M_{JJ}.

Noticing that each principal submatrix of an arc-arc weighted adjacency matrix is itself an arc-arc weighted adjacency matrix associated with an obvious subgraph, we shall develop the results in terms of an arc-arc weighted adjacency matrix M defined with respect to a fixed but arbitrary weighted digraph G with the understanding that these results apply readily to each of the principal submatrices of possibly another arc-arc weighted adjacency matrix. In particular, this digraph G is not necessarily the one arising from the spatial equilibrium model discussed in the last section.

Recall that a cycle in a digraph is a sequence of arcs connecting a node to itself. The directions of the arcs are irrelevant in the cycle. A <u>minimal cycle</u> is one which contains the minimum number of arcs. It is important to observe that if an arc is incident to two nodes in a minimal cycle, then the arc must be one of those in the cycle. Throughout the paper, all digraphs are simple.

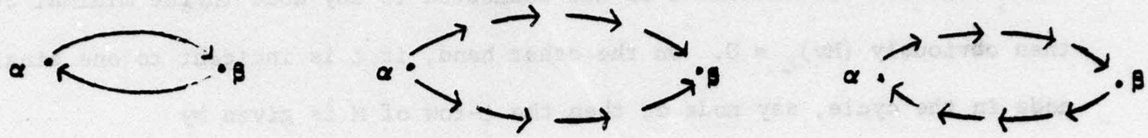

Figure 1: Examples of cycle

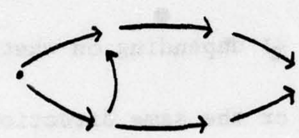

Figure 2: The outer cycle is not minimal

Proposition 3. If the digraph G contains a cycle, then its associated arc-arc weighted adjacency matrix M is singular.

Proof. It suffices to exhibit a nonzero vector v such that Mv = 0. Choose a minimal cycle in G and let $A_1, A_2, \ldots, A_L$ be the arcs in the cycle. Fix the direction of one of these arcs, say A_1. Define a vector $v = (v_k)$ as follows

$$v_k = \begin{cases} 1 & \text{if } k = A_i \text{ for some } i \text{ and if } A_i \text{ and } A_1 \text{ are oriented in the same direction} \\ -1 & \text{if } k = A_i \text{ for some } i \text{ and if } A_i \text{ and } A_1 \text{ are oriented in the opposite direction} \\ 0 & \text{otherwise.} \end{cases}$$

We claim that $Mv = 0$. Consider a specific component, say $(Mv)_\ell$ with ℓ denoting a certain arc. If the arc ℓ is not connected to any node in the minimal cycle, then obviously $(Mv)_\ell = 0$. On the other hand, if ℓ is incident to one single node in the cycle, say node α, then the ℓ-row of M is given by

$$\begin{array}{ccccccccccccc} & A_1 & A_2 & \dots & A_t & \dots & A_{t'} & \dots & A_{L-1} & A_L & \\ [\ \dots & 0 & 0 & \dots & s_1 b_\alpha & \dots & s_2 b_\alpha & \dots & 0 & 0 & \dots\ ] \end{array}$$

where A_t and $A_{t'}$ are the two arcs in the cycle which are incident to node α and where $s_1 (s_2)$ is equal to ± 1 depending on whether the arcs ℓ and $A_t (A_{t'})$ are oriented in the opposite or the same direction. If both pairs of arcs (ℓ and A_t) and (ℓ and $A_{t'}$) are oriented in the same way (i.e., if $s_1 = s_2$), then the arcs A_t and $A_{t'}$ must be oriented in the opposite direction (i.e., then $v_{A_t} = - v_{A_{t'}}$); on the other hand, if the two pairs of arcs (ℓ and A_t) and (ℓ and $A_{t'}$) are oriented differently (i.e., if $s_1 = - s_2$), then A_t and $A_{t'}$ must be oriented in the same direction (i.e., then $v_{A_t} = v_{A_{t'}}$). Consequently, in either case, we have

$$(Mv)_\ell = (s_1 v_{A_t} + s_2 v_{A_{t'}}) b_\alpha = 0 .$$

Finally, if ℓ is incident to two nodes in the cycle, then it must coincide with one of the L arcs by minimality of the cycle. Let's suppose $\ell = A_t = (\alpha,\beta)$. Then the ℓ-row of M is given by

$$\begin{array}{ccccccccccccccc} & A_1 & A_2 & \dots & A_t & \dots & A_{u_1} & \dots & A_{u_2} & \dots & A_{L-1} & A_L & \\ [\ \dots & 0 & 0 & \dots & (b_\alpha + b_\beta) & \dots & s_1 b_\alpha & \dots & s_2 b_\beta & \dots & 0 & 0 & \dots\ ] \end{array}$$

where A_{u_1} and A_{u_2} are the two arcs adjacent to A_t in the cycle and where $s_1(s_2)$ is equal to ± 1 depending on whether A_t and A_{u_1} (A_{u_2}) are oriented in the opposite or the same direction. By using a similar argument, we may deduce

$$(Mv)_\ell = (b_\alpha + b_\beta)v_{A_t} + s_1 b_\alpha v_{A_{u_1}} + s_2 b_\beta v_{A_{u_2}} = 0 .$$

Consequently, M is in fact singular. Q.E.D.

Given a nonsingular principal submatrix M_{11} of a square matrix M, there exists a permutation matrix P such that

$$P^T M P = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} .$$

The <u>Schur complement of</u> M_{11} <u>in</u> M is the matrix $M_{22} - M_{21}M_{11}^{-1}M_{12}$. Moreover, the Schur determinental formula (see Cottle [3] e.g.) is

$$\det M = \det M_{11} \times \det (M_{22} - M_{21}M_{11}^{-1}M_{12}) .$$

The next proposition is a partial converse of the last one.

<u>Proposition 4</u>. If the digraph G is a tree, then the arc-arc weighted adjacency matrix M is nonsingular.

<u>Proof</u>. We use induction on $n(\geq 2)$ the number of nodes in G. The assertion is obvious for $n = 2$. Suppose that it is true for a tree with n nodes. Consider a tree G with n + 1 nodes. Since G is a tree, it contains an end

node, say α. Let A be the only arc in G incident to α and let β be the other end point of the arc A. Let G' be the subtree obtained from G by deleting the node α and the arc A. Assign the same set of weights to all nodes in G' except for the node β where we assign the weight $1/(b_\alpha^{-1} + b_\beta^{-1})$. Let M' be the arc-arc weighted adjacency matrix obtained from the (weighted) digraph G'. By induction hypothesis, M' is nonsingular. We now show that M' is precisely the Schur complement of the diagonal entry M_{AA} in the original matrix M. With no loss of generality, we may assume that

$$M = \begin{pmatrix} M_{AA} & M_{A2} \\ M_{2A} & M_{22} \end{pmatrix}$$

where M_{A2} is given by

$$M_{A2} = [0 \dots 0 \; b_\beta \dots b_\beta \; -b_\beta \dots -b_\beta \; 0 \dots 0]$$

with the plus (minus) b_β appearing in those arcs (i.e., columns) which are incident to the node β and oriented in the opposite (same) direction as the arc A. Moreover, the principal submatrix of M_{22} corresponding to those arcs which are incident to the node β is given by

$$b_\beta \left[\begin{array}{c|c} E & -E \\ \hline -E & E \end{array}\right] + \Sigma$$

where E is the matrix of 1's and where Σ is the diagonal matrix whose diagonal entries are the weights of those nodes (except for α) which are adjacent to the node β. The same principal submatrix of $-M_{21}M_{AA}^{-1}M_{12}$ is given by

$$\frac{-b_\beta^2}{b_\alpha + b_\beta} \left[\begin{array}{c|c} E & -E \\ \hline -E & E \end{array}\right] .$$

In fact, all entries of the matrix $-M_{21}M_{AA}^{-1}M_{12}$ are zero except for those of this principal submatrix. It then follows that all entries of the Schur complement $M_{22}-M_{21}M_{AA}^{-1}M_{12}$ are the same as the corresponding ones of M_{22}, except for those in the principal submatrix which is equal to

$$\left(b_\beta - \frac{b_\beta^2}{b_\alpha + b_\beta}\right)\left[\begin{array}{c|c} E & -E \\ \hline -E & E \end{array}\right] + \Sigma = \frac{1}{b_\alpha^{-1} + b_\beta^{-1}}\left[\begin{array}{c|c} E & -E \\ \hline -E & E \end{array}\right] + \Sigma .$$

It is now obvious that $M_{22} - M_{21}M_{AA}^{-1}M_{12} = M'$. Since both M_{AA} and its Schur complement are nonsingular, it follows from the Schur determinental formula that so is M. This completes the inductive step and the proof. Q.E.D.

Remark. The above two propositions can also be proved by using Proposition 2 and the tree-property of the basis matrix of a linear transhipment problem (see Dantzig [4]).

Corollary 5. If the digraph G is a forest, then the arc-arc weighted adjacency matrix M is nonsingular.

Proof: It is sufficient to observe that such a matrix is block diagonal with each diagonal block corresponding to a tree. The corollary then follows immediately from Proposition 4. Q. E. D.

Corollary 6. Let M be the arc-arc weighted adjacency matrix associated with the (weighted) tree G. Then

$$\det M = (\Pi b_\gamma) \times (\Sigma b_\gamma^{-1})$$

where both the product and the summation range over all the nodes in G.

Proof. We use induction on the number n of nodes in G. The formula is certainly correct for n = 2. Suppose that it is true for a tree with n nodes. Consider now a tree G with n+1 nodes. Using the same notations as in the proof of Proposition 4, we have by the Schur determinental formula,

$$\det M = (b_\alpha + b_\beta) \times \det M'$$

Now the induction hypothesis implies that

$$\det M' = \prod_{\gamma \neq \alpha,\beta} b_\gamma \times 1/(b_\alpha^{-1} + b_\beta^{-1}) \times \sum_{\text{all } \gamma} b_\gamma^{-1}$$

From these two equations, the desired formula for det M follows readily.

Q. E. D.

Combining Corollary 5 and Proposition 3. we obtain the theorem below which characterizes the nonsingularity of an arc-arc weighted adjacency matrix associated with a (weighted) digraph.

<u>Theorem 7</u>. Let G be a digraph with positive weights on the nodes. Then the arc-arc weighted adjacency matrix is nonsingular if and only if the graph G is a forest.

Let $A = (\alpha,\beta)$ be a fixed but arbitrary arc in a tree G. We say that a node γ which is different from α and β, is <u>on level 1 with respect to the arc</u> A if γ is incident to A. Inductively, given a node γ not on level $k - 2$, we say that it is <u>on level k with respect to the arc A</u> if it is adjacent to one of the nodes on level $k - 1$. The nodes α and β are considered to be on level 0. Given two nodes γ_1 and γ_2 on levels k_1 and k_2 respectively with $k_1 < k_2$, we say that γ_2 is a <u>descendent</u> of γ_1 if the nodes in the unique path connecting node γ_1 to node γ_2 are on strictly descending levels. By convention, we consider a node as a descendent of itself. We shall denote by D_γ the set of descendents of the node γ. Notice that D_α and D_β partition the set of nodes in G. We call the number $\sum_{\sigma \in D_\gamma} b_\sigma^{-1}$

modified weight of the node γ, and denote it by $\bar{b}_{\gamma}$.

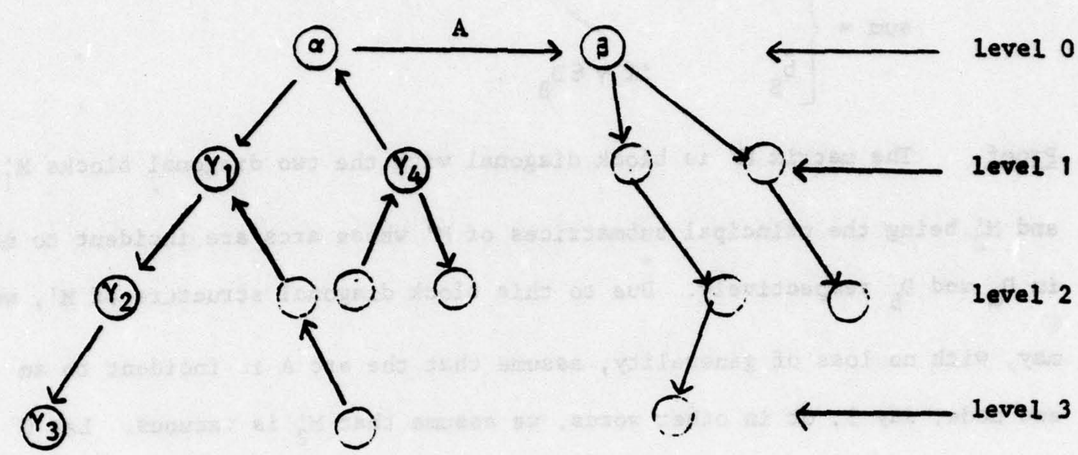

Figure 3: Different levels in a tree

Note: The node γ_3 is a decendent of γ_1 but not of γ_4 .

The two propositions below are fundamental to the solution of a system of linear equations involving an arc-arc weighted adjacency matrix.

Proposition 8. Let G be a weighted tree with M being its arc-arc weighted adjacency matrix. Let $A = (\alpha,\beta)$ be any arc in G. Let $M_{\cdot A}$ be the A-column of M with the diagonal entry deleted and let M' be the principal submatrix of M with both the A-row and the A-column deleted. Then the vector $(M')^{-1}M_{\cdot A}$ is given by the following: if $k \neq A$ is the arc (δ,γ) with $\gamma \in D_{\delta}$, then

$$((M')^{-1}M_{\cdot A})_k = \begin{cases} \bar{b}_{\gamma}/\text{sum} & \text{if k and A are oriented in the opposite direction} \\ -\bar{b}_{\gamma}/\text{sum} & \text{if k and A are oriented in the same direction} \end{cases}$$

where

$$\text{sum} = \begin{cases} \bar{b}_\alpha & \text{if } \gamma \in D_\alpha \\ \bar{b}_\beta & \text{if } \gamma \in D_\beta \end{cases} .$$

<u>Proof</u>. The matrix M' is block diagonal with the two diagonal blocks M_1' and M_2' being the principal submatrices of M' whose arcs are incident to nodes in D_α and D_β respectively. Due to this block diagonal structure of M', we may, with no loss of generality, assume that the arc A is incident to an end node, say β; or in other words, we assume that M_2' is vacuous. Let G' be the subtree obtained from G by deleting the arc A and the node β. We may further assume that M' has the form

$$M' = \begin{pmatrix} M_{11}' & M_{12}' \\ M_{21}' & M_{22}' \end{pmatrix}$$

where M_{11}' is the principal submatrix of M' whose arcs are incident to α. With respect to this partitioning, we may write

$$M_{1A} = b_\alpha \begin{pmatrix} M_{1A}' \\ 0 \end{pmatrix}$$

where M_{1A}' is a vector of plus and minus ones such that $(M_{1A}')_\ell = 1(-1)$ if the arcs ℓ and A are oriented in the opposite (same) direction. If we write $(M')^{-1}M_{1A} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ according to the above partitioning of M', we may easily deduce

$$(M_{11}' - M_{12}'(M_{22}')^{-1} M_{21}')x_1 = b_\alpha M_{1A}' .$$

By the proof of Proposition 4 and an inductive argument, it is not difficult to show that the Schur complement $(M'_{11} - M'_{12}(M'_{22})^{-1}M'_{21})$ is the arc-arc weighted adjacency matrix associated with the subtree G" obtained from G' by deleting all the arcs not incident to the node α and all the nodes not adjacent to α. The weight of a node σ in the subtree G" is equal to $(\bar{b}_\sigma)^{-1}$ if $\sigma \neq \alpha$ and equal to b_α if $\sigma = \alpha$. In fact, this Schur complement is given explicitly by

$$M'_{11} - M'_{12}(M'_{22})^{-1}M_{21} = \Sigma + b_\alpha M'_{1A}(M'_{1A})^T$$

where Σ is the diagonal matrix whose diagonal entries are the inverses of the modified weights of the nodes adjacent to α in G". According to Sherman-Morrison-Woodbury formula (see Householder [8, p. 124] e.g.), we obtain

$$x_1 = b_\alpha \left[\Sigma^{-1} - \frac{b_\alpha \Sigma^{-1} M'_{1A}(M'_{1A})^T \Sigma^{-1}}{b_\alpha (M'_{1A})^T \Sigma^{-1} M'_{1A}} \right] M'_{1A}$$

which gives

$$x_1 = \Sigma^{-1} M'_{1A} / \bar{b}_\alpha \, .$$

This establishes the desired formula for $((M')^{-1}M_{1A})_k$ provided that k is incident to node α. To establish the formula for k not incident to α, we evaluate x_2. It is not difficult to see that M'_{22} is again a block diagonal matrix with each diagonal block being the principal submatrix whose arcs are incident to nodes in D_σ for some node $\sigma \neq \alpha$ in the tree G". Again we may assume that M'_{22} consists of just one single block with all the arcs incident to a certain descendent of a specific note $\sigma \neq \alpha$ in G". We then have

$$M'_{22}x_2 = -M'_{21}x_1 = -b_\alpha x_{\alpha\sigma}\begin{pmatrix} M'_{2A'} \\ 0 \end{pmatrix}$$

where A' is the arc connecting α and σ, say A' = (α,σ), and $M'_{2A'}$ is defined in a fashion similar to M'_{1A} above. By decomposing $x_2 = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$ with x_{21} corresponding to those arcs incident to the node σ and using the same argument, we may easily deduce

$$x_{21} = -x_{\alpha\sigma} * (\Sigma')^{-1} M'_{2A'} / \bar{b}_\sigma$$

where Σ' is the diagonal matrix whose diagonal entries are the inverses of the modified weights of the nodes adjacent to σ. Substituting $x_{\alpha\sigma}$ into the above expression for x_{21} immediately yields the desired formula for $((M')^{-1}M_{1A})_k$ for k incident to σ. The proof of the proposition can now be completed by repeatedly using the above argument to all branches of the tree G. Q. E. D.

<u>Proposition 9</u>. Let G,M,A,M_{1A} and M' be as in Proposition 8. Then

$$M_{AA} - (M_{1A})^T (M')^{-1} M_{1A} = (\bar{b}_\alpha)^{-1} + (\bar{b}_\beta)^{-1}$$

<u>Proof</u>. To simplify the notations, we assume, as we did earlier, that the arc A is incident to the end node β. Then by Corollary 6 and the Schur determinental formula, we have

$$M_{AA} - (M_{1A})^T (M')^{-1} M_{1A} = \det M/\det M'$$

$$= \left(\prod_{\gamma \in V} b_\gamma\right) \times \sum_{\gamma \in V} b_\gamma^{-1} \Bigg/ \left[\left(\prod_{\substack{\gamma \in V \\ \gamma \neq \beta}} b_\gamma\right) \times \sum_{\substack{\gamma \in V \\ \gamma \neq \beta}} b_\gamma^{-1}\right]$$

$$= b_\beta + (\bar{b}_\alpha)^{-1}$$

as desired. Q. E. D.

Using the same notations as in Proposition 8, we consider the solution of the system of linear equations $Mx = d$ for some given vector d. We may assume with no loss of generality that M' is a leading principal submatrix of M. Partitioning the vectors x and d accordingly we may write

$$\begin{pmatrix} M' & M_{1A} \\ (M_{1A})^T & M_{AA} \end{pmatrix} \begin{pmatrix} x' \\ x_A \end{pmatrix} = \begin{pmatrix} d' \\ d_A \end{pmatrix}$$

which implies

$$(M_{AA} - (M_{1A})^T (M')^{-1} M_{1A}) x_A = d_A - (M_{1A})^T (M')^{-1} d' \tag{4a}$$

and

$$x' = (M')^{-1} d' - (M')^{-1} M_{1A} x_A . \tag{4b}$$

Combined with Propositions 8 and 9, these latter two formulas show how the two vectors x and $(M')^{-1}d'$ can be efficiently computed from each other. In fact, suppose that $(M')^{-1}d'$ is known, then Proposition 8 shows that $(M_{1A})^T(M')^{-1}$

can be obtained by identifying the descendents of each node. Together with Proposition 9, formula (4a) yields the component x_A readily. Substituting x_A and using Proposition 8 again, we may easily compute x' from (4b). Conversely, if x is known, then $(M')^{-1}d'$ can be obtained readily from (4b) as well.

4. SPECIALIZATION OF THE PARAMETRIC PRINCIPAL PIVOTING ALGORITHM

In this section, we specialize the parametric principal pivoting algorithm to solve the linear complementarity problem (2) arising from the spatial equilibrium model. Our purposes are (1) to show that the problem can be solved by performing the 1×1 diagonal pivots exclusively and (2) to derive from this result and those established in the last section an efficient specialized algorithm for solving the problem. To achieve these, we first state two preliminary results having to do with the application of the parametric principal pivoting algorithm for solving a general parametric linear complementarity problem with a symmetric positive semi-definite matrix.

Lemma 10. Let M be any symmetric positive semi-definite matrix. Consider the solution of the parametric linear complementarity problem (3) by Algorithm 1. Let J denote the index set of the basic z-variables at each iteration. Then M_{JJ} is nonsingular. Moreover, if a 2×2 block pivot occurs at a certain iteration and if r and s are the two critical indices obtained in Steps 1 and 3 of the algorithm, then $r \notin J$ and $s \in J$.

The two assertions contained in this lemma are rather well-known in the theory of pivotal algebra. Their proofs are thus omitted. (The authors are grateful to Professor I. Kaneko for pointing out this fact.)

Corollary 11. Let I be the index set of the currently nonbasic z-variables. If the maximum ratio in Step 1 of Algorithm 1 does not occur at the nonbasic index $i \in I$, then the variable z_i can not become basic at the next pivot.

Proof. Under the given assumption, the only possible way for z_i to become basic is for i equal to the critical index s. But Lemma 10 asserts that this is impossible. Q. E. D.

With these two general results, we proceed to solve the linear complementarity problem (2). Recall that the lower limit $\underline{\lambda}$ of the parameter λ is zero. Combining Lemma 10 with Theorem 7, we obtain

<u>Theorem 12</u>. Throughout the solution process, the set of basic flow variables can not contain a cycle.

In [4], Glassey showed that a solution to (1) can be found which contains no transshipment of flows. The theorem below extends this result.

<u>Theorem 13</u>. Let the parametric vector p be chosen such that all components are equal. Then throughout the solution process, there is no transshipment of flows.

We need three lemmas to establish the theorem. From now on, we let the vector p be chosen as stated in the theorem.

<u>Lemma 14</u>. Let J be the index set of the currently basic flow variables. Suppose that $k = (\alpha,\beta) \in J$. Then the nonbasic variable z_ℓ with $\ell = (\beta,\alpha)$ can not become basic at the next iteration.

<u>Proof</u>. Notice that Theorem 12 implies that $\ell \notin J$. The ℓ-component of the current q-vector before the next pivot is given by

$$\bar{q}_\ell = q_\ell - M_{\ell J} M_{JJ}^{-1} q_J .$$

It is easy to verify that

$$(M_{\ell J} M_{JJ}^{-1})_j = \begin{cases} -1 & \text{if } j = k \\ 0 & \text{otherwise} . \end{cases}$$

Therefore, we have

$$\bar{q}_\ell = q_\ell + q_k = c_{\alpha\beta} + c_{\beta\alpha} \geq 0 .$$

Hence, according to Corollary 11 and the termination criteria, the variable z_ℓ can not become basic at the next iteration. Q. E. D.

<u>Lemma 15.</u> Let J be as in Lemma 14. Suppose that $k_1 = (\alpha,\beta)$ and $k_2 = (\alpha,\gamma)$ with $\beta \neq \gamma$ are in J. Then the nonbasic variables z_{ℓ_1} and z_{ℓ_2} where $\ell_1 = (\beta,\gamma)$ and $\ell_2 = (\gamma,\beta)$ can not become basic at the next iteration.

<u>Proof</u>. Notice that Theorem 12 implies that $\ell_1 \notin J$ and $\ell_2 \notin J$. The ℓ_1-component of the current q-vector is given by

$$\bar{q}_{\ell_1} = q_{\ell_1} - M_{\ell_1 J}(M_{JJ})^{-1} q_J .$$

It is not difficult to show that (cf. the proof of Proposition 3 e.g.)

$$(M_{\ell_1 J} M_{JJ}^{-1})_j = \begin{cases} -1 & \text{if } j = k_1 \\ 1 & \text{if } j = k_2 \\ 0 & \text{otherwise.} \end{cases}$$

Consequently, we have

$$\bar{q}_{\ell_1} = q_{\ell_1} + (q_{k_1} - q_{k_2}) = c_{\alpha\beta} + c_{\beta\gamma} - c_{\alpha\gamma} \geq 0$$

where the last inequality follows from the triangle inequality of the transportation costs. Hence according to Corollary 11 and the termination criteria, the variable z_{ℓ_1} can not become basic at the next iteration. Similarly, we may establish the same conclusion for z_{ℓ_2} . Q. E. D.

<u>Lemma 16</u>. Let J be as in Lemma 14. Suppose that J contains no transshipment of flows. If the arc ℓ is such that $J \cup \{\ell\}$ contains a cycle, then the nonbasic variable z_ℓ can not become basic at the next iteration. In fact, the ℓ-component of the current q-vector is nonnegative.

<u>Proof</u>. Theorem 12 implies that J contains no cycle and that $\ell \notin J$. Suppose that the cycle in $J \cup \{\ell\}$ contains an even number of arcs. By the above two lemmas, we may assume that this number is at least four. Notice that ℓ must be one of these arcs and that there are precisely two arcs adjacent to each arc in the cycle. Let $\ell = (\beta,\gamma)$ and let the two arcs adjacent to ℓ be $k_1 = (\alpha_1,\delta_1)$ and $k_2 = (\alpha_2,\delta_2)$. Because of the even number, there are two cases: ($\beta = \alpha_1$ and $\gamma = \delta_2$) or ($\gamma = \alpha_1$ and $\beta = \delta_2$). Consider the first case, namely $\beta = \alpha_1$ and $\gamma = \delta_2$ (cf. Figure 4). The ℓ-components of the current q- and p-vector are given by

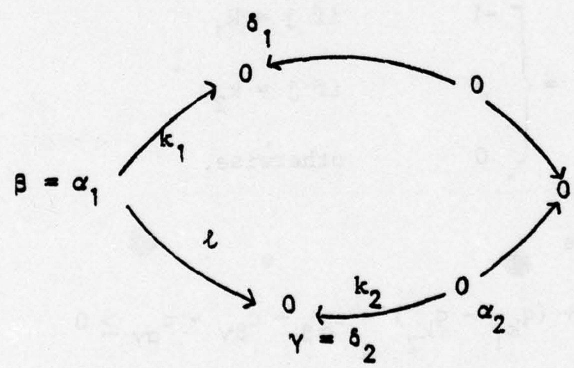

Figure 4. An even cycle with no transshipment

$$\bar{q}_\ell = q_\ell - M_{\ell J}M_{JJ}^{-1}q_J \quad \text{and} \quad \bar{p}_\ell = p_\ell - M_{\ell J}M_{JJ}^{-1}p_J \; .$$

By the proof of Proposition 3, one can easily deduce that

$$(M_{\ell J}M_{JJ}^{-1})_j = \begin{cases} 0 & \text{if } j \text{ is not in the cycle} \\ 1 & \text{if } j \text{ is in the cycle and oriented in the opposite direction as } \ell \\ -1 & \text{if } j \text{ is in the cycle and oriented in the same direction as } \ell \; . \end{cases}$$

From this and an easy calculation, we may deduce

$$\bar{q}_\ell = \sum_j c_j - \sum_i c_i \qquad \text{and } \bar{p}_\ell = 0$$

where the first and second sums in the expression for $\bar{q}_\ell$ range over those arcs in the cycle which are oriented in the same and opposite directions as ℓ respectively. Notice that $\bar{p}_\ell = 0$ follows from the fact that the cycle contains an even number of arcs. Consequently, according to the ratio test in Step 1 of Algorithm 1 and Corollary 11, z_ℓ will not become basic at the next pivot. Moreover, we must have $\bar{q}_\ell \geq 0$ because the non-negativity of the component $\bar{q}_\ell + \lambda\bar{p}_\ell$ is maintained throughout the algorithm.

Consider the second case, namely $\gamma = \alpha_1$ and $\beta = \delta_2$ (cf. Figure 5). By the same token, we may similarly deduce

$$\bar{q}_\ell = \sum_j c_j - \sum_i c_i$$

where the two summations are ranging over the same sets of arcs as before. By the triangle inequality, it follows that

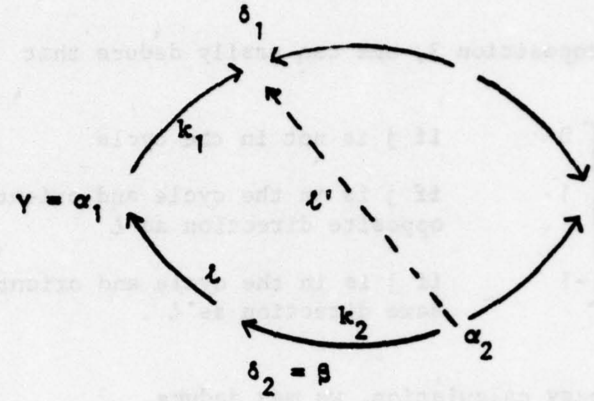

Figure 5. An even cycle with transshipment

$\bar{q}_\ell \geq \bar{q}_{\ell'} \geq 0$, where $\ell' = (\alpha_2, \delta_1)$. The last inequality follows from the first case. Consequently, the variable z_ℓ can not become basic at the next pivot. Similarly, we may establish the same assertion in the case where the cycle is odd. Q. E. D.

<u>Proof of Theorem 13</u>. We use induction on the number of iterations. The assertion is obviously true at the first iteration. Suppose that it is true after t iterations and the algorithm is entering its (t + 1)st iteration. Let J be the index set of the currently basic flow variables. By induction hypothesis J contains no transshipment of flows. Moreover, by Theorem 12, J contains no cycle as well. It suffices to show if $k = (\alpha, \beta) \in J$, then all variables z_ℓ with $\ell = (\beta, \gamma)$ or $\ell = (\gamma, \alpha)$ can not become basic at the next

pivot. To be specific, we let $\ell = (\beta,\gamma)$. By Lemma 14, we may assume that $\gamma \neq \alpha$. Moreover, by Lemma 15, we may assume that $i = (\alpha,\gamma) \notin J$. Finally by Lemma 16, we may assume that $J \cup \{\ell\}$ contains no cycle as well. The current i- and ℓ-components of the p-vector are given by

$$\bar{p}_i = p_i - M_{iJ}M_{JJ}^{-1}p_J \quad \text{and} \quad \bar{p}_\ell = p_\ell - M_{\ell J}M_{JJ}^{-1}p_J .$$

Subtracting the two equations gives

$$\begin{aligned}\bar{p}_\ell - \bar{p}_i &= (M_{iJ} - M_{\ell J})M_{JJ}^{-1}p_J \\ &= M_{kJ}M_{JJ}^{-1}p_J = p_k .\end{aligned}$$

Similarly, we may deduce

$$\bar{q}_i - \bar{q}_\ell = c_{\alpha\gamma} - (c_{\beta\gamma} + c_{\alpha\beta}) \leq 0 .$$

Hence, it follows that

$$- \bar{q}_\ell\bar{p}_i + \bar{q}_i\bar{p}_\ell = (-\bar{q}_\ell + \bar{q}_i)\bar{p}_i + \bar{q}_i p_k .$$

Suppose that $\bar{p}_i \leq 0$. If the maximum ratio occurs at the arc ℓ such that $- \bar{q}_\ell / \bar{p}_\ell > 0$, then we have

$$0 < - \bar{q}_\ell \leq - \bar{q}_i$$

so that $\bar{q}_i + \lambda\bar{p}_i < 0$ for nonnegative values of λ. But this contradicts the fact that $\bar{q}_i + \lambda\bar{p}_i$ is kept nonnegative throughout the algorithm. Therefore, if $\bar{p}_i \leq 0$, then either the algorithm terminates with the arc ℓ remaining nonbasic or else the maximum ratio cannot occur at the arc ℓ. In either case, the theorem is proved.

Suppose now $\bar{p}_i > 0$. Then we have

$$- \bar{q}_\ell \bar{p}_i + \bar{q}_i \bar{p}_\ell \leq \bar{q}_i p_k .$$

Now, if the left-hand term is nonnegative, then so is $\bar{q}_i$ which then implies that $\bar{q}_\ell \geq 0$. In this case the variable z_ℓ can not become basic in the next iteration. On the other hand, if the left-hand term is negative, then we would have

$$- (\bar{q}_\ell / \bar{p}_\ell) < - (\bar{q}_i / \bar{p}_i)$$

unless $\bar{p}_\ell \leq 0$. Consequently, the maximum ratio will not occur at the arc ℓ. The desired conclusion therefore follows from Corollary 11.

Q. E. D.

We may now state and prove our principal result.

Theorem 17. Let the parametric vector p be chosen to be the vector of ones. Then throughout the solution process, all pivots are 1 x 1 diagonal. In other words, all the diagonal pivot entries are nonzero.

Proof. We use induction on the number of pivot iterations. The assertion is certainly true at the first pivot because all diagonal entries of M are positive. Suppose that the assertion is true for t iterations and the algorithm is entering its (t + 1)st iteration. Let J be the index set of basic flow variables at the end of the t-th iteration. Suppose now that the maximum ratio occurs at the critical index r such that the corresponding diagonal pivot entry is zero. Then $r \notin J$ and the matrix

$$\begin{pmatrix} M_{JJ} & M_{Jr} \\ M_{rJ} & M_{rr} \end{pmatrix}$$

is singular. Consequently the set of arcs in $J \cup \{r\}$ must contain a cycle by Theorem 7. Since J contains no transshipment of flows, Lemma 16 implies that the current critical value of λ is nonpositive. In other words, the algorithm will terminate without performing further pivots. Q. E. D.

Summarizing the discussion, we now present the promised algorithm for solving the linear complementarity problem (2) arising from the spatial equilibrium model. The algorithm is a straightforward specialization of Algorithm 2 to this application.

The Algorithm.

Step 0 (Initialization) Let $J = \emptyset$. Determine

$$\lambda = \max \{ -(a_\alpha - a_\beta + c_{\alpha\beta}) : \alpha \neq \beta , \alpha , \beta \in V \} .$$

If $\lambda \leq 0$, terminate. A solution is given by $x = 0$. Otherwise, let $k = (\alpha,\beta)$ be a maximizing arc. Let

$$J' = \begin{cases} J & \text{if } k \notin J \\ J \setminus \{k\} & \text{if } k \in J . \end{cases}$$

Step 1 (Search for descendents) Identify the descendents of the nodes α and β which are incident to arcs in J'. Determine the modified weights of these descendents (including the nodes α and β as well). Compute the vector $M^{-1}_{J'J'}M_{J'k}$ according to Proposition 7*.

Step 2 (Updating of basic components) Set

$$(\bar{p}_k , \bar{q}_k) = \begin{cases} (\bar{p}_k , \bar{q}_k) / (\bar{b}_\alpha^{-1} + \bar{b}_\beta^{-1}) & \text{if } k \notin J \\ (\bar{p}_k , \bar{q}_k) & \text{if } k \in J \end{cases}$$

* The set J' may be a forest but not a tree. The proposition is nevertheless applicable because obviously, $(M^{-1}_{J'J'}M_{J'k})_j = 0$ if the arc j is not connected to k. Hence we need to compute only those components which are connected to k.

and

$$(\bar{p}_{J'}, \bar{q}_{J'}) = (\bar{p}_{J'}, \bar{q}_{J'}) - M^{-1}_{J'J'} M_{J'k} (\bar{p}_k , \bar{q}_k) .$$

Step 3 (Updating of basic index set) Set

$$J = \begin{cases} J \cup \{k\} & \text{if } k \notin J \\ J \setminus \{k\} & \text{if } k \in J \end{cases}$$

Step 4 (Updating of nonbasic components) Let $I = J^c$. Compute

$$(\bar{p}_I , \bar{q}_I) = (p_I , q_I) - M_{IJ}(\bar{p}_J , \bar{q}_J) .$$

Step 5 (Ratio test) Determine

$$\lambda = \max\{\max\{-\bar{q}_i / \bar{p}_i : \bar{p}_i > 0 , i \in I\}, \max\{-\bar{q}_j / \bar{p}_j : \bar{p}_j < 0 , j \in J\}\}.$$

If $\lambda \leq 0$, terminate. A solution is given by $\begin{pmatrix} x_I \\ x_J \end{pmatrix} = \begin{pmatrix} 0 \\ -\bar{q}_J \end{pmatrix}$. Otherwise, let

$k = (\alpha,\beta)$ be a maximizing arc. Go to Step 1.

According to the third remark at the end of Section 2, the proposed algorithm will compute a set of equilibrium flows in a finite number of steps.

Some Computational Remarks. The updating of the nonbasic components in Step 4 is easy to carry out. Indeed, for each nonbasic arc, it suffices to determine the basic arcs which are adjacent to it. This provides an efficient way to multiply the product $M_{IJ}(\bar{q}_J , \bar{p}_J)$ from which the updating can be achieved trivially. Furthermore, the relationship

$$(\bar{q}_i , \bar{p}_i) + (\bar{q}_{i'} , \bar{p}_{i'}) = (c_i + c_{i'} , 2p_i)$$

for nonbasic arcs $i = (\alpha,\beta)$ and $i' = (\beta,\alpha)$, can be used to reduce the computational effort in this step. The ratio test in Step 5 (and in Step 0 as well) can be somewhat simplified by noting that $\bar{q}_i \leq 0 \Rightarrow \bar{q}_{i'} \leq 0$ if i and i' are as just mentioned. This implication implies that in carrying out the test, at most one of the two arcs i and i' need to be considered. Finally, the various results established in this section guarantee that a number of arcs will not become basic at the next iteration. They can thus be ignored in the test.

5. COMPUTATIONAL EXPERIENCE

We have implemented the proposed algorithm for solving some randomly generated problems. The data are generated as follows: The a_i and b_i are random numbers lying in the interval [0,50] and [0,20] respectively. The cost c_{ij} is given by $c_{ij} = d_i + d_j$ where d_i is equal to 10 if i belongs to a certain random index set and is a random number in [0,20] otherwise. The reason for generating the costs in this way is to ensure that the triangle inequality will be satisfied. The code is written in FORTRAN and the runs are performed on a DEC-20 computer at Carnegie-Mellon University. The results are summarized in the table below. The times are exclusive of input and output.

# nodes	# arcs	# basic arcs	# pivots	CPU time (in sec.)	time/pivot (in sec.)
40	1,760	24	56	15.850	0.28
80	6,320	45	65	82.456	1.27
120	14,280	70	176	732.421	4.16
160	25,440	108	420	5,682.318	13.53

Notice that the number of arcs is equal to N(N-1) where N is the number of regions. Moreover, the number of basic arcs can not exceed N-1.

Looking at the CPU times, one could argue that the proposed algorithm is perhaps not performing as efficiently as it should be. However, one has to realize that although these problems could be considered as having a sparse matrix (the arc-arc matrix of an N-node problem has $2N(N-1)^2$ nonzero entries, i.e., density = $\frac{200}{N}$ %), they are still too big for the direct application of most currently existing general-purpose quadratic programming and/or linear complementarity algorithms. These times are roughly proportional to $10^{-5} \times n^4$, a figure which is really not big in comparison with the size of the arc-arc matrix. The performance of the algorithm can be further improved by taking advantage of an important consequence of the result concerning the no transhipment of flows. This consequence enables one to fix at least half of the flow variables at the value 0 and thus reduces the size of the problem by at least one half. It is stated and proved in the theorem below.

Theorem 17. If $a_\alpha - a_\beta + c_{\alpha\beta} > 0$, then the arc $k = (\alpha, \beta) \notin J$ in each iteration.

Proof. Since there is no transhipment of flows, it is easy to see that the basis matrix M_{JJ} is nonnegative in each iteration. If such an arc k were in J at some iteration, then since complementarity is preserved in the algorithm, we would have

$$0 = q_k + \lambda p_k + M_{kJ} x_J > 0$$

which is a contradiction. Q. E. D.

It should be pointed out that this result has not been incorporated in the coding of the algorithm

Conclusion. In this paper, we have proposed and implemented a parametric linear complementarity approach for the computation of equilibrium prices in an economic spatial model addressed by Glassey and others. In essence, the proposed algorithm is a specialized version of the parametric principal pivoting algorithm applied to solve the linear complementarity problem arising from the model. Like Glassey's algorithm, ours can be stated and implemented based entirely on the underlying digraph of the model.

As mentioned in the introduction, the model treated in this paper is a simplification of some other models of a more general nature. In the near future, we plan to extend the technique used here to treat various generalizations of the model and will report our findings elsewhere.

APPENDIX

We shall show in the sequel that Glassey's Algorithm A1 [4] fails to achieve internal equilibrium in certain situations. Following the notations in the reference, suppose (e_1, i_1) is an incoming arc, and $E = \{e_1, \ldots, e_m\}$ and $I = \{i_1, \ldots, i_n\}$ are the coalitions containing e_1 and i_1 respectively. Furthermore, for simplicity, we suppose that there is no outgoing arc.

Thus, for the new prices $\bar{p}_{e_j}$ and $\bar{p}_{i_j}$ of nodes $e_j \in E$ and $i_j \in I$ calculated according to the algorithm, we have

$$\bar{p}_{e_j} = p_{e_j} + \delta_e \, , \; \bar{p}_{i_j} = p_{i_j} + \delta_i \text{ and } \bar{p}_{e_1} + c_{e_1 i_1} - \bar{p}_{i_1} = 0$$

where p_{e_j} and p_{i_j} are respectively the prices of the nodes $e_j \in E$ and $i_j \in I$ before updating.

Since originally, respective nodes in E and I have achieved internal equilibrium, we have for all $k \neq \ell$

$$p_{e_k} + c_{e_k e_\ell} - p_{e_\ell} \geq 0 \quad \text{and} \quad p_{i_k} + c_{i_k i_\ell} - p_{i_\ell} \geq 0$$

which imply

$$p_{e_k} + c_{e_k e_\ell} - p_{e_\ell} + \delta_e - \delta_e \geq 0 \quad \text{and} \quad p_{i_k} + c_{i_k i_\ell} - p_{i_\ell} + \delta_i - \delta_i \geq 0$$

which in turn imply that the new coalition consisting of E and I will have internal equilibrium within themselves.

Now if the new coalition formed by $E \cup I$ attains internal equilibrium, then we have for $k = 1,\ldots,m$ and $\ell = 1,\ldots,n$

$$\bar{p}_{e_k} + c_{e_k i_\ell} - \bar{p}_{i_\ell} \geq 0 \quad \text{and} \quad \bar{p}_{i_\ell} + c_{i_\ell e_k} - \bar{p}_{e_k} \geq 0$$

or equivalently,

$$p_{e_k} + c_{e_k i_\ell} - p_{i_\ell} + \delta_e - \delta_i \geq 0 \quad \text{and} \quad p_{i_\ell} + c_{i_\ell e_k} - p_{e_k} + \delta_i - \delta_e \geq 0$$

But $\delta_i - \delta_e = p_{e_1} + c_{e_1 i_1} - p_{i_1}$. (See [4]). Hence

$$(p_{e_k} + c_{e_k i_\ell} + p_{i_1}) - (p_{i_\ell} + p_{e_1} + c_{e_1 i_1}) \geq 0$$

for all $k = 1,\ldots,m$; $\ell = 1,\ldots,n$.

It is now obvious that the above inequality does not necessarily hold in general, and thus, internal equilibrium is not guaranteed.

Similarly, it can be shown that if an arc goes out of the basis simultaneously, then the same situation occurs.

In what follows, we provide a counterexample showing how the algorithm fails to maintain internal equilibrium.

The data are given as follows (for seven nodes):

$a_1 = 19.481866$, $a_2 = 60.873993$, $a_3 = 10.845196$, $a_4 = 6.172308$, $a_5 = 63.791228$, $a_6 = 67.071617$, $a_7 = 37.919311$; $b_1 = 7.324636$, $\beta_1 = 0.136526$, $b_2 = 3.225784$, $\beta_2 = 0.310002$, $b_3 = 1.884832$, $\beta_3 = 0.530554$, $b_4 = 7.132555$, $\beta_4 = 0.140202$, $b_5 = 0.487533$, $\beta_5 = 2.051144$, $b_6 = 3.922178$, $\beta_6 = 0.254960$, $b_7 = 5.383146$, $\beta_7 = 0.185765$, and $c_{ij} = |i - j| = c_{ji}$, for all i,j.

We choose (e,i) to become basic whenever $\overline{c_{ei}} = \min\limits_{i,j} \{p_i + c_{ij} - p_j\}$.

Following the procedure described by Glassey [5], we have:

Cycle 1. Form coalition $C_1 = (4\ 6)$

Cycle 2. Form coalition $C_2 = (3\ 5)$

Cycle 3. Form coalition $C_3 = (1\ 2)$

Cycle 4. Form new coalition $C_2 = (735)$. Step 2c gives $\theta_{35} = 104.497$ and $x_{75} = 2.322544$. By algorithm, no arc is going out.

Cycle 5. Form new coalition $C_4 = C_1 \cup C_2 = (34567)$. Step 2 gives new equilibrium prices as: $\bar{p}_7 = 49.516340$, $\bar{p}_6 = 52.516340$, $\bar{p}_5 = 51.516340$, $\bar{p}_4 = 50.516340$, $\bar{p}_3 = 49.516340$. Step 2c gives $\theta_{75} = 34.600461$, $\theta_{35} = 109.527072$ and $\theta_{46} = 8.257801$ and $x_{45} = 2.506109$. By algorithm, no arc is going out.

Notice that at this point, internal equilibrium is not achieved in coalition $C_4 = (34567)$.

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This paper presents a parametric linear complementarity technique for the computation of equilibrium prices in a single commodity spatial model. We first reformulate the model as a linear complementarity problem and then apply the parametric principal pivoting algorithm for its solution. This reformulation leads to the study of an "arc-arc weighted adjacency matrix" associated with a simple digraph having weights on the nodes. Several basic properties of such a matrix are derived. Using these properties, we show how the parametric principal pivoting algorithm can be greatly simplified in this application. Finally, we report some computational experience.

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